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Large Deviations for the Fermion Point Process Associated with the Exponential Kernel

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For the fermion point process on the whole complex plane associated with the exponential kernel $e^{z\bar{w}}$, we show the central limit theorem for the random variable $\xi(D_r)$, the number of points inside the ball D_r of radius r, as $r \to \infty$ and we establish the large deviation principle for the random variables $\{r^{-2}\xi(D_r), r > 0\}$.

KEY WORDS: fermion point process, determinantal point process, Ginibre ensemble, exactly solvable, large deviations, exponential kernel

1. INTRODUCTION

Let $\{\zeta_n\}_{n\geq 0}$ be a sequence of independent standard complex Gaussian random variables and $\{c_n\}_{n\geq 0}$ a (deterministic) sequence of complex numbers. The random power series of the form $X(z) = \Sigma_{n=0}^{\infty} c_n \zeta_n z^n$ is a typical example of a Gaussian analytic function. The set of its zeros has been widely studied in the contexts of both mathematics and physics (cf.[1−4]). Recently it is shown in ref. 5 that the zeros of the Gaussian analytic function $X_{\text{hyp}}(z) = \Sigma_{n=0}^{\infty} \zeta_n z^n$ in the unit disk $U \subset \mathbf{C}$ turn out to be the fermion (or determinantal) point process μ_{Berg} in U associated with the Bergman kernel $K_{\text{Berg}}(z, w) = \pi^{-1}(1 - z\bar{w})^{-2}$, which is the reproducing kernel of the L^2-space of analytic functions in the unit disk. The Gaussian analytic (entire) function $X_{\text{flat}}(z) = \Sigma_{n=0}^{\infty} (n!)^{-1/2} \zeta_n z^n$, whose covariance is given by $e^{z\bar{w}}$, is studied as well as $X_{\text{hyp}}(z)$; the terms “hyp” and “flat” indicate the symmetry (cf.[1,3,4]). The zeros of X_{flat} become a point process on the whole complex plane, say ν_{flat}, which is invariant under translations and rotations; its mean measure is $\pi^{-1} m(dz)$, where $m(dz)$ is the Lebesgue measure on $\mathbf{C}$. In ref. 6, the decay rate of the hole probability that there are no points inside the ball D_r of radius r is given

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as follows: for some positive constants c and C,

$$-c \leq \liminf_{r\to\infty} \frac{1}{r^4} \log \nu_{\text{flat}}(\xi(D_r) = 0) \leq \limsup_{r\to\infty} \frac{1}{r^4} \log \nu_{\text{flat}}(\xi(D_r) = 0) \leq -C,$$

where $\xi(D_r)$ is the number of points inside D_r. If we consider the Poisson point process Π on the whole complex plane with the same mean measure $\pi^{-1}m(dz)$ as that of ν_{flat}, it is obvious that $\Pi(\xi(D_r) = 0) = -r^2$, so the hole probability of ν_{flat} has faster decay than that of Poisson Π. Here we consider the fermion point process $\mu_{\exp}$ on the whole complex plane associated with the exponential kernel $K(z, w) = e^{z\bar{w}}$, which is the reproducing kernel of the L^2-space of entire functions with respect to the standard complex Gaussian measure. It is the counterpart of μ_{Berg} for the whole complex plane and, moreover, it appears in the limit of the eigenvalue point processes of the Ginibre complex matrix ensemble.[7] In the physics literature, $\mu_{\exp}$ arises from the two-dimensional one-component plasma or jellium model at a special temperature and it is well-known to be exactly solvable (cf.[8]). As is shown in Proposition 3.1, $\mu_{\exp}$ has the same symmetry and mean measure as those of ν_{flat} and Π; the resemblance between ν_{flat} and $\mu_{\exp}$ is discussed, for example, in ref. 9. The exact order r^4 for the hole probability has been captured for the two-dimensional one-component plasma and also large deviations problem has been discussed under several scalings in ref. 8. In the present paper, by emphasizing the structure of fermion point processes, we show the full large deviation principle for the random variables $\{r^{-2}\xi(D_r), r > 0\}$, where $r^2 = E\xi(D_r)$, and compute exactly the rate function. In particular, the decay order of the hole probability turns out to be the same r^4 as that of ν_{flat} but not Π.

Theorem 1.1. *Let $\mu_{\exp}$ be the fermion point process on the whole complex plane associated with the integral operator on $L^2(\boldsymbol{C}, \lambda)$ with exponential kernel $K(z, w) = e^{z\bar{w}}$, where λ is the standard complex Gaussian measure. Then*

$$\lim_{r\to\infty} \frac{1}{r^4} \log \mu_{\exp}(\xi(D_r) = [ar^2]) = -I(a), \tag{1.1}$$

where $[x]$ is the largest integer which does not exceed x and

$$I(a) = \begin{cases} \dfrac{1}{4}|2a^2 \log a - (a-1)(3a-1)| & a \geq 0 \\ \infty & a < 0. \end{cases}$$

Moreover, for every measurable set Γ,

$$\begin{aligned} -\inf_{x\in\Gamma^\circ} I(x) &\leq \liminf_{r\to\infty} \frac{1}{r^4} \log \mu_{\exp}(r^{-2}\xi(D_r) \in \Gamma) \\ &\leq \limsup_{r\to\infty} \frac{1}{r^4} \log \mu_{\exp}(r^{-2}\xi(D_r) \in \Gamma) \leq -\inf_{x\in\overline{\Gamma}} I(x). \end{aligned}$$

Remark 1.2.

(1) *The rate function $I(a)$ is given by the integral of the rate function for the sum of Poisson random variables with mean 1, i.e.,*

$$I(a) = \left| \int_1^a (1 - x + x \log x)\, dx \right|.$$

(2) *The asymptotics of the Laplace transforms is given as*

$$\lim_{r\to\infty} \frac{1}{r^2} \log E e^{\alpha\xi(D_r)} = \alpha, \quad \forall \alpha \in \boldsymbol{R}, \tag{2}$$

where E stands for the expectation with respect to μ_{exp}. It might be said that the mean-field approximation is valid in this model.

We also give the asymptotics of the variance of $\xi(D_r)$ with respect to μ_{exp} and show the central limit theorem.

Theorem 1.3. *The variance of the number of points in D_r is given by*

$$\begin{aligned}\mathrm{var}(\xi(D_r)) &= \frac{r}{\pi} \int_0^{4r^2} (1 - x/4r^2)^{1/2} x^{-1/2} e^{-x}\, dx \\ &\sim \frac{r}{\sqrt{\pi}}\end{aligned}$$

as $r \to \infty$. In particular, the central limit theorem holds:

$$\frac{\pi^{1/4}(\xi(D_r) - r^2)}{\sqrt{r}} \xrightarrow{d} N(0, 1).$$

Remark 1.4. *The variance var($\xi(D_r)$) corresponds to that of the net electronic charge Q_{D_r} in equilibrium classical infinitely extended Coulomb systems, for which the variance formula has been obtained in refs. 8 and 10 and it shows that the variance behaves like*

$$\mathrm{var}(Q_{D_r}) \sim -\frac{S_{\partial D_r}}{\pi} \int d^2\boldsymbol{r}\, r s(r),$$

where $S_{\partial D_r}$ is the volume of the boundary of D_r and $s(r)$ is the charge-charge correlation function. Under the appropriate normalization, one can see that both results coincide. We give a proof of Theorem 1.3 based on the Bernoulli structure of $\xi(D_r)$ or Q_{D_r} (Proposition 2.2).

In Sec. 2 and 3, we recall the basics of fermion point processes and study some basic properties of μ_{exp}. In Section 4, we give proofs of Theorem 1.1, Theorem 1.3 and (1.2).

2. FERMION POINT PROCESSES

In this section, we recall some well-known facts for fermion point processes.

Let R be a locally compact Hausdorff space with countable basis and $\mathcal{B}(R)$ the topological Borel σ-field. We fix a Radon measure $\lambda(dx)$ on $(R, \mathcal{B}(R))$. The configuration space $Q = Q(R)$ is the totality of non-negative integer-valued Radon measures on R and it is given the topology which is generated by the functions $Q \ni \xi \mapsto \xi(A) \in \mathbf{R}$ for every $A \in \mathcal{B}(R)$.

We can summarize the existence and uniqueness result for a fermion point process associated with kernel $K(x, y)$ as follows.[11,12]

Theorem 2.1. *Let K be a self-adjoint integral operator on $L^2(R, \lambda)$ with kernel $K(x, y)$. Suppose that Spec$(K) \subset [0, 1]$ and K is of locally trace class, i.e., for any compact set $\Lambda \subset R$, $K_\Lambda = I_\Lambda K I_\Lambda$ is of trace class. Then there exists a unique Borel probability measure μ_K on Q such that for any non-negative bounded measurable function f with compact support*

$$\int_Q \mu_K(d\xi) \exp\left(-\int_R \xi(dx) f(x)\right) = \mathrm{Det}(I - K_\phi), \tag{2.1}$$

where $\phi = 1 - e^{-f}$, $K_\phi = \sqrt{\phi} K \sqrt{\phi}$ and Det$(I - K_\phi)$ is the Fredholm determinant of the integral operator K_ϕ. Moreover, the n-th correlation measure λ_n is given by

$$\lambda_n(dx_1 \dots dx_n) = \det(K(x_i, x_j))_{i,j=1}^n \lambda^{\otimes n}(dx_1 \dots dx_n). \tag{2.2}$$

The resultant point process μ_K is called a fermion point process after[13] or often called a determinantal point process associated with the kernel K. General properties and convergence theorems for fermion point processes are found in (cf.[11,12]).

For a compact set $\Lambda \subset R$, we focus on the random variable $\xi(\Lambda)$, the number of points in Λ. We recall some basic results for $\xi(\Lambda)$.

Proposition 2.2. *The probability of the event $\{\xi(\Lambda) = k\}$ is given by the formula: for $k = 0, 1, 2, \dots,$*

$$\mu_K(\xi(\Lambda) = k) = \sum_{\substack{I \subset \{0,1,2,\dots\} \\ |I| = k}} \prod_{i \in I} \kappa_i \prod_{j \in I^c} (1 - \kappa_j). \tag{2.3}$$

where $\kappa_n \in [0, 1], n = 0, 1, 2, \ldots$ *are the eigenvalues of* K_Λ. *Moreover, if* $X_0, X_1, \ldots$ *are independent* $\{0, 1\}$*-valued random variables each of which obeys the Bernoulli distribution* $Be(\kappa_n)$, $n = 0, 1, 2, \ldots$, *then*

$$\xi(\Lambda) \stackrel{d}{=} \sum_{n=0}^{\infty} X_n. \tag{2.4}$$

Proof: Observing that $1 - e^{-f} = (1 - e^{-\alpha})I_\Lambda$ when $f = \alpha I_\Lambda (\alpha \geq 0)$ and $\kappa_n \in [0, 1]$ by the assumption of K, we see from (2.1) that

$$E\left[e^{-\alpha\xi(\Lambda)}\right] = \mathrm{Det}(I - (1 - e^{-\alpha})K_\Lambda) = \prod_{n=0}^{\infty}(1 - \kappa_n + e^{-\alpha}\kappa_n) \tag{2.5}$$

$$= \prod_{n=0}^{\infty} Ee^{-\alpha X_n} = E\left[e^{-\alpha \Sigma_{n=0}^{\infty} X_n}\right],$$

which implies (2.4). The formula (2.3) follows immediately from (2.4). □

Proposition 2.3. *The mean and variance of* $\xi(\Lambda)$ *are given as*

$$E\xi(\Lambda) = \mathrm{Tr}\, K_\Lambda, \quad \mathrm{var}\,(\xi(\Lambda)) = \mathrm{Tr}\, K_\Lambda(I - K_\Lambda).$$

Proof: Since $\xi(\Lambda) \stackrel{d}{=} \sum_{n=0}^{\infty} X_n$ from Proposition 2.2, we get

$$E\xi(\Lambda) = \sum_{n=0}^{\infty} EX_n = \sum_{n=0}^{\infty} \kappa_n, \quad \mathrm{var}\,(\xi(\Lambda)) = \sum_{n=0}^{\infty} \mathrm{var}(X_n) = \sum_{n=0}^{\infty} \kappa_n(1 - \kappa_n).$$

□

The central limit theorems for fermion point processes have been studied (cf.[14–16]). By noticing the fact that the number of points inside the ball is the sum of independent random variables, one can give the following proof, which is also given in ref. 17 along this way.

Proposition 2.4. *Let* $\{\Lambda_n\}_{n\geq 1}$ *be an increasing sequence of compact subsets such that var*$(\xi(\Lambda_n))$ *goes to* ∞. *Then,* $\xi(\Lambda_n)/E\xi(\Lambda_n) \to 1$ *a.s., and* $(\xi(\Lambda_n) - E\xi(\Lambda_n))/\sqrt{\mathrm{var}(\xi(\Lambda_n))}$ *converges in distribution to the standard normal distribution* $N(0, 1)$.

Proof: We remark that $\mathrm{var}(\xi(\Lambda)) \leq E\xi(\Lambda)$. The law of large numbers follows from the standard argument. For the central limit theorem, one can easily show the convergence of the characteristic functions by using the identity (2.4) and the following fact: let $\alpha_n(t) \in \mathbf{C}$, $n = 0, 1, 2, \ldots$ be a sequence of complex numbers satisfying (i) $\sup_{t>0} \Sigma_{n=0}^{\infty} |\alpha_n(t)| < \infty$, (ii) $\Sigma_{n=0}^{\infty} \alpha_n(t) \to \alpha$ as $t \to \infty$ and (iii) $\sup_{n\geq 0} |\alpha_n(t)| \to 0$ as $t \to \infty$. Then $\Sigma_{n=0}^{\infty} \log(1 + \alpha_n(t)) \to \alpha$. □

3. FERMION POINT PROCESS ASSOCIATED WITH THE EXPONENTIAL KERNEL

Let $R = \mathbf{C}$ and $\lambda(dz)$ be the standard complex Gaussian measure on $\mathbf{C}$, i.e.,

$$\lambda(dz) = \frac{1}{\pi} e^{-|z|^2} m(dz),$$

where $m(dz)$ is the Lebesgue measure on $\mathbf{C}$. Let $L^2(\mathbf{C}, \lambda)$ be the L^2-space over $\mathbf{C}$ with inner product

$$\langle f, g \rangle = \int_{\mathbf{C}} f(z)\overline{g(z)}\lambda(dz).$$

We consider the closed subspace of entire functions of $L^2(\mathbf{C}, \lambda)$ and denote it by $A^2(\mathbf{C}, \lambda)$. Remark that the entire functions

$$\varphi_n(z) = \frac{z^n}{\sqrt{n!}}, n = 0, 1, 2, \ldots \tag{3.1}$$

form a complete orthonormal basis of $A^2(\mathbf{C}, \lambda)$. Let $K : L^2(\mathbf{C}, \lambda) \to L^2(\mathbf{C}, \lambda)$ be the integral operator with kernel $K(z, w) = e^{z\bar{w}}$, which is the reproducing kernel of $A^2(\mathbf{C}, \lambda)$ in the sense that $\langle f, K(\cdot, w)\rangle = f(w)$ for any $w \in \mathbf{C}$. Since

$$K(z, w) = \sum_{n=0}^{\infty} \varphi_n(z)\overline{\varphi_n(w)},$$

the operator K defines the orthogonal projection from $L^2(\mathbf{C}, \lambda)$ onto $A^2(\mathbf{C}, \lambda)$. In particular, its eigenvalues are 0 or 1 of infinite multiplicities.

Let μ_{exp} be the fermion point process on $\mathbf{C}$ associated with the exponential kernel $K(z, w) = e^{z\bar{w}}$ with respect to λ. We remark that the integral kernel is given by

$$\tilde{K}(z, w) = e^{-|z-w|^2/2+\sqrt{-1}\,\mathrm{Im}\, z\bar{w}} = e^{z\bar{w}-|z|^2/2-|w|^2/2}$$

with respect to $\pi^{-1}m$ instead of λ. Although the kernel $\tilde{K}$ itself is not invariant under translations, the point process μ_{exp} is invariant under translations and rotations. Indeed, we have the following:

Proposition 3.1. *The first and second correlation measure of μ_{exp} are given by*

$$\lambda_1(dz) = \frac{1}{\pi} m(dz), \quad \lambda_2(dz_1 dz_2) = \frac{1}{\pi^2}\left(1 - e^{-|z_1-z_2|^2}\right) m^{\otimes 2}(dz_1\, dz_2),$$

respectively. In general, the n-th correlation measure is invariant under translations and rotations in the whole complex plane.

Proof: By the formula (2.2), we see that the first correlation density with respect to the Lebesgue measure m is given by

$$\pi \frac{d\lambda_1}{dm}(z) = K(z,z)e^{-|z|^2} = 1,$$

and the second one

$$\begin{aligned}\pi^2 \frac{d\lambda_2}{dm^{\otimes 2}}(z_1,z_2) &= \det(K(z_i,z_j))_{i,j=1}^2 e^{-(|z_1|^2+|z_2|^2)} \\ &= \left(e^{|z_1|^2+|z_2|^2} - e^{z_1\bar{z}_2+\bar{z}_1 z_2}\right)e^{-(|z_1|^2+|z_2|^2)} \\ &= 1 - e^{-|z_1-z_2|^2}.\end{aligned}$$

For the n-th correlation measure,

$$\begin{aligned}\pi^n \frac{d\lambda_n}{dm^{\otimes n}}(z_1,\dots,z_n) &= \det(K(z_i,z_j))_{i,j=1}^n e^{-\Sigma_{i=1}^n |z_i|^2} \\ &= \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^n e^{z_i \bar{z}_{\sigma(i)}} \cdot e^{-\Sigma_{i=1}^n |z_i|^2} \\ &= \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) \exp(f_\sigma(z_1,\dots,z_n)),\end{aligned}$$

where $f_\sigma(z_1,\dots,z_n) = \sum_{i=1}^n z_i(\overline{z_{\sigma(i)} - z_i})$. For $a, b \in \mathbf{C}$, $|a| = 1$, we consider the transformation $z_i \mapsto az_i + b$, and then

$$\begin{aligned}f_\sigma(az_1+b,\dots,az_n+b) &= \sum_{i=1}^n (az_i+b)\overline{((az_{\sigma(i)}+b)-(az_i+b))} \\ &= \sum_{i=1}^n (z_i + b\bar{a})(\overline{z_{\sigma(i)} - z_i}) \\ &= f_\sigma(z_1,\dots,z_n).\end{aligned}$$

The last equality follows from the equality $\Sigma_{i=1}^n(\overline{z_{\sigma(i)} - z_i}) = 0$ for every $\sigma \in S_n$ and it implies the second assertion. □

Lemma 3.2. *Let $D_r = \{|z| \le r\}$ and $K_r = K_{D_r} : L^2(\boldsymbol{C}, \lambda) \to L^2(\boldsymbol{C}, \lambda)$. Then the eigenvalues of K_r except 0 are given by*

$$\kappa_n = \frac{1}{n!} \int_0^{r^2} t^n e^{-t} dt = \sum_{k=n+1}^{\infty} \frac{r^{2k} e^{-r^2}}{k!}, \quad n = 0, 1, 2, \ldots, \tag{3.2}$$

and the corresponding eigenfunctions are φ_n, $n = 0, 1, 2, \ldots$ defined in (3.1). In particular, K_r is of trace class.

Proof: Putting $\varphi_{n,r} = \varphi_n I_{D_r} / \|\varphi_n I_{D_r}\|$, we obtain an orthonormal system $\{\varphi_{n,r}\}_{n=0}^{\infty}$. By the definition of K_r,

$$K_r(z, w) = \sum_{n=0}^{\infty} \varphi_n(z) I_{D_r}(z) \overline{\varphi_n(w) I_{D_r}(w)} = \sum_{n=0}^{\infty} \|\varphi_n I_{D_r}\|^2 \varphi_{n,r}(z) \overline{\varphi_{n,r}(w)}.$$

Then the eigenvalues of K_r are $\|\varphi_n I_{D_r}\|^2 = 0, 1, \ldots$, and their eigen-functions are $\varphi_{n,r}$, $n = 0, 1, \ldots$ All we have to do is compute the norm $\|\varphi_n I_{D_r}\|^2$. Since $|Z|^2$ is the exponential random variable with mean 1 when Z is the standard complex Gaussian random variable, we get

$$\begin{aligned} \kappa_n = \|\varphi_n I_{D_r}\|^2 &= \int_{D_r} \frac{|z|^{2n}}{n!} \lambda(dz) \\ &= \frac{1}{n!} E[|Z|^{2n}; |Z|^2 \le r^2] = \frac{1}{n!} \int_0^{r^2} t^n e^{-t} dt \end{aligned}$$

In particular, Tr $K_r = r^2$. □

Remark 3.3. *Let $X_0, X_1, \ldots$ be a sequence of independent, identically distributed exponential random variables with mean 1 and Y_{r^2} the Poisson random variable with mean r^2. Then we get two useful expressions of κ_n:*

$$\kappa_n = \kappa_n(r) = P(S_n \le r^2) \tag{3.3}$$

$$= P(Y_{r^2} \ge n + 1), \tag{3.4}$$

where $S_n = X_0 + \cdots + X_n$. So it is obvious that $\kappa_n = \kappa_n(r)$ is monotone decreasing in n and monotone increasing in r. Moreover, if f in Theorem 2.1 is a function of $|z|$, then the eigenvalues of K_ϕ are given by

$$\kappa_n(f) = E[\phi(\sqrt{S_n})]$$

for $n = 0, 1, \ldots$, where $\phi = 1 - e^{-f}$. This fact is essentially pointed out as covariance structure for the eigenvalue (finite) point process of Ginibre's complex random matrix in. refs.18 and 19.

Remark 3.4. *In the same way as above, one can discuss the fermion point process associated with the Bergman kernel K_{Berg}. Let $R = U = \{|z| < 1\}$ and $\lambda(dz) = m(dz)$, the Lebesgue measure on R. We consider the L^2-space $L^2(U)$ with respect to the inner product $\langle f, g\rangle_U = \int_U f(z)\overline{g(z)}m(dz)$ and the closed subspace of $L^2(U)$ of analytic functions, i.e., $A^2(U) = \{f : U \to \boldsymbol{C}; ||f||_U < \infty, f \text{ is analytic}\}$. If we put $\psi_n(z) = \pi^{-1/2}(n+1)^{1/2}z^n$, $\{\psi_n\}_{n\geq 0}$ forms an orthonormal basis of $A^2(U)$. Then the integral operator K on $L^2(U)$ with the Bergman kernel is the orthogonal projection onto $A^2(U)$ and the eigenvalues (except 0) of the restriction $K_r = K_{D_r}$, $0 \leq r < 1$, are given by $\kappa_n = r^{2n+2}$, $n = 0, 1, \ldots,$ and the corresponding eigenfunction is ψ_n. Let $X_0, X_1, \ldots$ be a sequence of independent, identically distributed random variables whose common law is the uniform distribution on $[0, 1]$, $T_n = \max_{0\leq i\leq n} X_i$ and Z_{r^2} the geometric distribution with parameter r^2. Then we get similar expressions of κ_n as $\kappa_n = P(T_n \leq r^2) = P(Z_{r^2} \geq n+1)$; moreover, for a function f of $|z|$, $\kappa_n(f) = E[\phi(\sqrt{T_n})]$ as in the above remark.*

4. PROOFS OF THE THEOREMS

In this section, we give proofs of (1.2), Theorem 1.1 and Theorem 1.3. Throughout this section, we write $\rho = r^2$, for simplicity.

Proof of Theorem 1.3: It is obvious by (3.4) that $E\xi(D_r) = EY_\rho = \rho$. Setting $p_k = P(Y_\rho = k)$, we have

$$\sum_{n=0}^{\infty} \kappa_n^2 = \sum_{n=0}^{\infty} \sum_{k,\ell=n+1}^{\infty} p_k p_\ell = \sum_{k,\ell=1}^{\infty} k \wedge \ell \cdot p_k p_\ell$$
$$= E[Y_\rho \wedge Y'_\rho],$$

where Y'_ρ is an independent copy of Y_ρ. By Proposition 2.3, we have

$$\mathrm{var}(\xi(D_r)) = \mathrm{T}_r K_r(I - K_r) = EY_\rho - E[Y_\rho \wedge Y'_\rho] = E[X_\rho; X_\rho \geq 0],$$

where $X_\rho = Y_\rho - Y'_\rho$. Let $\varphi_\rho(\theta)$ be the characteristic function of X_ρ. Then we see that

$$\varphi_\rho(\theta) = E\exp(i\theta X_\rho) = \exp(\rho(e^{i\theta} - 1))\exp(\rho(e^{-i\theta} - 1))$$
$$= \exp\left(-4\rho\sin^2\frac{\theta}{2}\right)$$

and that

$$\mathrm{var}(\xi(D_r)) = E[X_\rho; X_\rho \geq 0] = \sum_{n=1}^{\infty} nP(X_\rho = n)$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{n}{2\pi} \int_0^{2\pi} \varphi_\rho(\theta) e^{-in\theta} d\theta = \sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_0^{2\pi} \varphi_\rho'(\theta) e^{-in\theta} d\theta \\
&= \lim_{N\to\infty} \frac{1}{2\pi i} \int_0^{2\pi} \varphi_\rho'(\theta) \frac{e^{-i\theta}(1-e^{-iN\theta})}{1-e^{-i\theta}} d\theta \\
&= \frac{1}{2\pi i} \int_0^{2\pi} \varphi_\rho'(\theta) \frac{e^{-i\theta}}{1-e^{-i\theta}} d\theta
\end{aligned}
$$

by Riemann-Lebesgue's theorem. The real part of the R.H.S. is the desired expression:

$$
\begin{aligned}
\mathrm{var}(\xi(D_r)) &= \frac{r^2}{\pi} \int_0^{2\pi} \cos^2 \frac{\theta}{2} \cdot \exp\left(-4r^2 \sin^2 \frac{\theta}{2}\right) d\theta \\
&= \frac{r}{\pi} \int_0^{4r^2} (1 - x/4r^2)^{1/2} x^{-1/2} e^{-x} dx \ .
\end{aligned}
$$

The last integral converges to $\Gamma(1/2) = \sqrt{\pi}$ and so the asymptotics is obtained. The central limit theorem immediately follows from Proposition 2.4. $\square$

Lemma 4.1. *Let κ_n be as before. Then, for any $\beta \geq -1$,*

$$
\lim_{\rho\to\infty} \frac{1}{\rho} \left| \sum_{n=[\rho]}^{\infty} \log(1+\beta\kappa_n) \right| = \lim_{\rho\to\infty} \frac{1}{\rho} \left| \sum_{n=0}^{[\rho]} \log \kappa_n \right| = 0.
$$

Proof: First we note that

$$
\kappa_{[a\rho]} = P\big(S_{[a\rho]} \leq \rho\big) \to \begin{cases} 1 & 0 \leq a < 1, \\ 1/2 & a = 1, \\ 0 & a > 1. \end{cases} \tag{4.1}
$$

Then for any sufficiently large $\rho > 0, \kappa_n \leq 2/3$ for any $n \geq [\rho]$. For $\beta \geq -1$, there exists an $A = A_\beta > 0$ such that $|\log(1+\beta x)| \leq Ax (0 \leq x \leq 2/3)$. Then, for any sufficiently large ρ,

$$
\begin{aligned}
\frac{1}{\rho} \left| \sum_{n=[\rho]}^{\infty} \log(1+\beta\kappa_n) \right| &\leq \frac{A}{\rho} \sum_{n=[\rho]}^{\infty} P(Y_\rho \geq n+1) \\
&= \frac{A}{\rho} \sum_{k=[\rho]+1}^{\infty} (k - [\rho]) P(Y_\rho = k)
\end{aligned}
$$

$$
\begin{aligned}
&= A\left\{P(Y_\rho = [\rho]) + \frac{\rho - [\rho]}{\rho} P(Y_\rho \geq [\rho] + 1)\right\} \\
&\leq A\left\{\frac{\rho^{[\rho]} e^{-\rho}}{\sqrt{2\pi[\rho]}[\rho]^{[\rho]} e^{-[\rho]}} + \frac{\rho - [\rho]}{\rho}\right\} \\
&\to 0\,(\rho \to \infty)\,.
\end{aligned}
$$

Here we used Stirling's formula for the last inequality. In the same manner,

$$
\begin{aligned}
\frac{1}{\rho}\left|\sum_{n=0}^{[\rho]} \log \kappa_n\right| &= \frac{1}{\rho}\left|\sum_{n=0}^{[\rho]} \log(1 - (1 - \kappa_n))\right| \\
&\leq A\left\{P(Y_\rho = [\rho]) + \frac{[\rho] + 1 - \rho}{\rho} P(Y_\rho \leq [\rho])\right\} \\
&\to 0\,(\rho \to \infty)\,.
\end{aligned}
$$

□

Now we prove (1.2) by using Lemma 4.1.

Proof of (1.2): By (2.5) and Lemma 4.1, it suffices to show

$$
\lim_{\rho \to \infty} \frac{1}{\rho} \log E\left[e^{\alpha \xi(D_r)}\right] = \lim_{\rho \to \infty} \frac{1}{\rho} \sum_{n=0}^{[\rho]-1} \log(1 + \beta \kappa_n) = \alpha
$$

where $\beta = e^\alpha - 1$. When $\beta > 0$, the inequality

$$
\begin{aligned}
&(1 - \delta) \log\left(1 + \beta \kappa_{[(1-\delta)\rho]}\right) + (\delta - 2\rho^{-1}) \log\left(1 + \beta \kappa_{[\rho]-1}\right) \\
&\leq \frac{1}{\rho} \sum_{n=0}^{[\rho]-1} \log(1 + \beta \kappa_n) \leq \log(1 + \beta \kappa_0)
\end{aligned}
$$

holds for any $\delta > 0$. Since δ is arbitrary, letting $\rho \to \infty$, we get the assertion from (4.1). In the case of $-1 < \beta \leq 0$, one can show the assertion in the same way.

Proposition 4.2. *For any* $0 \leq a \leq b$,

$$
\lim_{\rho \to \infty} \frac{1}{\rho^2} \log \prod_{n=[a\rho]}^{[b\rho]} (1 - \kappa_n) = -\int_a^b J(x) \chi_{[0,1]}(x)\, dx \tag{4.2}
$$

and

$$\lim_{\rho\to\infty}\frac{1}{\rho^2}\log\prod_{n=[a\rho]}^{[b\rho]}\kappa_n = -\int_a^b J(x)\chi_{[1,\infty]}(x)dx\ , \tag{4.3}$$

where $J(x) = 1 - x + x\log x$ and $\chi_A(x)$ is the indicator function of A. In particular,

$$\lim_{\rho\to\infty}\frac{1}{\rho^2}\log\prod_{n=0}^{[a\rho]-1}\kappa_n\prod_{n=[a\rho]}^{\infty}(1-\kappa_n) = -\frac{1}{4}|2a^2\log a - (a-1)(3a-1)| \tag{4.4}$$

for any $a \geq 0$.

Proof: Here we show only (4.2) because (4.3) can be shown in the same way. Let S_n be as in (3.3). For any $0 \leq s < t < 1$, by the large deviations result for the sum of independent, identically distributed exponential random variables, we obtain

$$\begin{aligned}\frac{1}{\rho^2}\sum_{n=[s\rho]+1}^{[t\rho]}\log P(S_n \geq \rho) &\leq \frac{1}{\rho^2}([t\rho]-[s\rho])\log P(S_{[t\rho]} \geq \rho)\\ &\leq \frac{1}{\rho^2}([t\rho]-[s\rho])\log P\big(S_{[t\rho]} \geq t^{-1}[t\rho]\big)\\ &\to -(t-s)\cdot tI_1(t^{-1})\ ,\end{aligned} \tag{4.5}$$

where $I_1(x) = x - 1 - \log x$ is the rate function (cf.[20]). Similarly,

$$\begin{aligned}\frac{1}{\rho^2}\sum_{n=[s\rho]+1}^{[t\rho]}\log P(S_n \geq \rho) &\geq \frac{1}{\rho^2}([t\rho]-[s\rho])\log P(S_{[s\rho]+1} \geq \rho)\\ &\to -(t-s)\cdot sI_1(s^{-1})\ .\end{aligned} \tag{4.6}$$

Now putting $J(x) = xI_1(x^{-1}) = 1 - x + x\log x$, by the definition of Riemann's integral and Lemma 4.1, we get

$$\begin{aligned}\lim_{\rho\to\infty}\frac{1}{\rho^2}\sum_{n=[a\rho]}^{[b\rho]}\log(1-\kappa_n) &= \lim_{\rho\to\infty}\frac{1}{\rho^2}\sum_{n=[a\rho]}^{[b\rho]}\log P(S_n \geq \rho)\\ &= -\int_a^b J(x)\chi_{[0,1]}(x)\,dx\end{aligned}$$

for $0 \leq a \leq b$. A direct computation shows (4.4) from (4.2) and (4.3). □

Remark 4.3. *If one uses the expression (3.4) instead of (3.3), one obtains (4.5) and (4.6) as a direct consequence of the large deviations result for Poisson random variables; indeed, $J(x)$ is nothing but the rate function for them.*

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1: Since the lower bound is clear from (2.3) and (4.4), it suffices to show the inequality

$$\limsup_{\rho\to\infty} \frac{1}{\rho^2} \log \sum_{\substack{I\subset\{0,1,2,\dots\}\\ |I|=[a\rho]}} \prod_{i\in I} \kappa_i \prod_{i\in I^c}(1-\kappa_i) \le \lim_{\rho\to\infty} \frac{1}{\rho^2} \log \prod_{i=0}^{[a\rho]-1} \kappa_i \prod_{i=[a\rho]}^{\infty} (1-\kappa_i)$$

or, equivalently,

$$\limsup_{\rho\to\infty} \frac{1}{\rho^2} \log \sum_{\substack{I\subset\{0,1,2,\dots\}\\ |I|=[a\rho]}} \prod_{i\in I} h(\kappa_i) \le \lim_{\rho\to\infty} \frac{1}{\rho^2} \log \prod_{i=0}^{[a\rho]-1} h(\kappa_i)\,, \tag{4.7}$$

where $h(x) = x/(1-x)^{-1}$. Note that the non-negative function $h(x)$ on [0, 1) is monotone increasing, $h(1/2) = 1$ and $h(x) \le 3x$ on [0, 2/3].

First we consider the case where $0 \le a < 1$ and fix $M > 1$ large enough. Set

$$\mathcal{I}_{m,k} = \{I \subset \{0,1,2,\dots\}; |I| = m, |I \cap [0,[M\rho])^c| = k\}$$

for $k = 0, 1, \dots, m$. Then, for any sufficiently large $\rho > 0$, we have

$$\begin{aligned}
\sum_{\substack{I\subset\{0,1,2,\dots\}\\ |I|=m}} \prod_{i\in I} h(\kappa_i) &= \sum_{k=0}^{m} \sum_{I\in\mathcal{I}_{m,k}} \left(\prod_{i\in I\cap[0,[M\rho])} h(\kappa_i) \right) \left(\prod_{j\in I\cap[0,[M\rho])^c} h(\kappa_j) \right) \\
&\le \sum_{k=0}^{m} \left(\prod_{i=0}^{m-k-1} h(\kappa_i) \right) \binom{[M\rho]}{m-k} \cdot \frac{1}{k!} \left(\sum_{j=[M\rho]}^{\infty} h(\kappa_j) \right)^k \\
&\le \left(\prod_{i=0}^{[a\rho]-1} h(\kappa_i) \right) \binom{[M\rho]}{[a\rho]} \sum_{k=0}^{m} \frac{1}{k!} \left(\sum_{j=[M\rho]}^{\infty} h(\kappa_j) \right)^k \\
&\le \left(\prod_{i=0}^{[a\rho]-1} h(\kappa_i) \right) \left(\frac{e[M\rho]}{[a\rho]} \right)^{[a\rho]} e^{3\rho}
\end{aligned} \tag{4.8}$$

for every $m = 0, 1, \dots, [a\rho]$.

Next we consider the case where $a > 1$. Then, for any sufficiently large ρ,

$$\begin{aligned}\sum_{\substack{I\subset\{0,1,2,\ldots\}\\|I|=m}} \prod_{i\in I} h(\kappa_i) &= \sum_{n=m-1}^{\infty} \sum_{\substack{I\subset\{0,1,\ldots,n\}\\n\in I}} \prod_{i\in I} h(\kappa_i)\\ &\leq \left(\prod_{i=0}^{m-2} h(\kappa_i)\right) \sum_{n=m-1}^{\infty} \binom{n}{m-k} h(\kappa_n)\\ &\leq \left(\prod_{i=0}^{[a\rho]-2} h(\kappa_i)\right) \sum_{n=m-1}^{\infty} \binom{n}{m-1} 3\kappa_n\\ &= 3\left(\prod_{i=0}^{[a\rho]-2} h(\kappa_i)\right) \frac{\rho^m}{m!} \end{aligned} \tag{4.9}$$

for every $m \geq [a\rho]$. The last equality follows from the first equality in (3.2). From (4.8), (4.9), we can show (4.7) and hence (1.1).

Moreover, it is easy to see that

$$\mu_{\exp}(\xi(D_r) \leq a\rho) \leq ([a\rho]+1) \prod_{i=0}^{[a\rho]-1} \kappa_i \prod_{i=[a\rho]}^{\infty} (1-\kappa_i) \left(\frac{e[M\rho]}{[a\rho]}\right)^{[a\rho]} e^{3\rho}$$

for $0 \leq a < 1$, and

$$\mu_{\exp}(\xi(D_r) \geq a\rho) \leq 3 \prod_{i=0}^{[a\rho]-2} \kappa_i \prod_{i=[a\rho]-1}^{\infty} (1-\kappa_i) \cdot e^{\rho} P(Y_\rho \geq [a\rho])$$

for $a > 1$, from which the large deviation principle easily follows.

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